

Sugihara's Impossible Cylinder

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Kokichi Sugihara, a professor of engineering in Japan, has spent decades making objects look and move in seemingly impossible ways. This year Sugihara's video *Ambiguous Optical Illusion* received second prize in the Neural Correlate Society's best illusion of the year contest (<http://bit.ly/ambcylin>). It shows a variety of three-dimensional objects that look like one shape when viewed from the front but like a different shape in a mirror propped up behind it.

Sugihara's video became a viral sensation. It was viewed approximately 5 million times in the first week.

Are such shapes impossible? No, they just look that way. It is all about perspective. We show you the mathematics behind Sugihara's method for making one of the simpler shapes: one that looks like a circular cylinder from the front and a square cylinder in the mirror. And we give a template for you to make your own paper model (see figure 1).

Finding the Curve

The key to this illusion is that the top of the cylinder is not a planar curve. Figure 2 shows three views of the

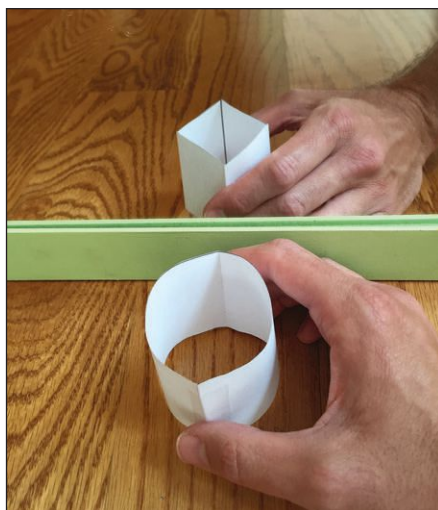


Figure 1. Sugihara's impossible cylinder made from paper.

closed curve that represents the top rim of Sugihara's shape. In one view, it looks as if we are looking at the unit circle in the xy -plane. In another, it looks like a square diamond. (Visit <http://bit.ly/29itVkk> for an interactive applet in which you can rotate this curve in space.)

To set this up mathematically, suppose we have two viewers looking down on a curve floating above the xy -plane (in the video, this second viewer is you, in the mirror). From the viewers' perspectives, it looks as if they are viewing the curves $y = f(x)$ and $y = g(x)$, respectively.

In our example, the two observed curves are the unit circle and the square passing through the points $(0, \pm 1)$ and $(\pm 1, 0)$, as shown in figure 3. We will have to break each of these shapes into different curves. We'll have $f(x) = 1 - |x|$ for half the square and $g(x) = \sqrt{1 - x^2}$ for half the circle. Then we take their negatives to obtain the other halves.

Now we add a z -axis. We assume the floating curve is given parametrically as $\mathbf{r}(t)$. One viewer thinks she is seeing the curve $(x, f(x), 0)$, and the other thinks he is seeing $(x, g(x), 0)$.

We would like the two viewers to look down on the curve at 45-degree angles on opposite sides of the xz -plane. We could give their exact locations as $(0, a, a)$ and $(0, -a, a)$ for some suitably large value a . But instead, we make the simplifying assumption that they are far enough away that they can treat the entire curve as if it is a single point; that is, the vectors from any

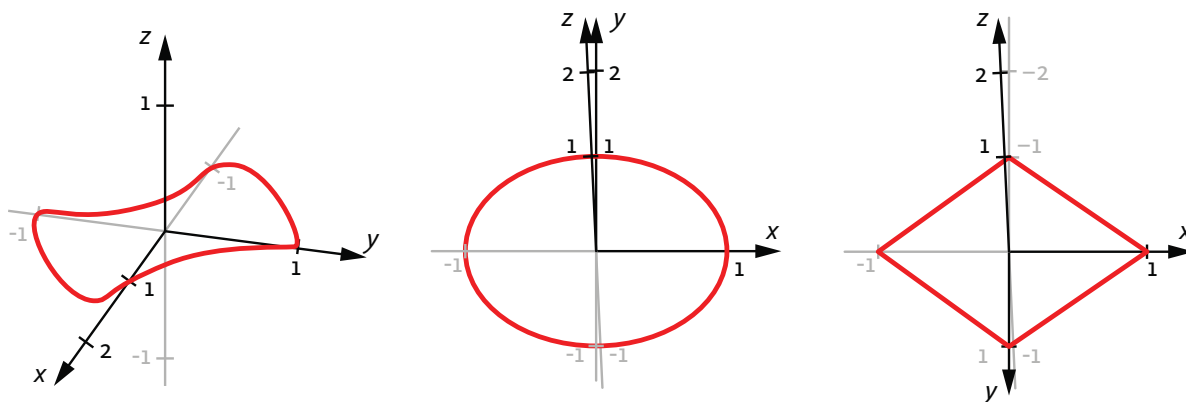


Figure 2. Three views of a closed curve. When we look down from the point $(0, -a, a)$, it looks like a circle in the xy -plane (center). When we look from $(0, a, a)$, it looks like a diamond in the xy -plane (right).

point on the curve to the viewers' eyes are parallel to $\langle 0, 1, 1 \rangle$ and $\langle 0, -1, 1 \rangle$, respectively. See figure 4.

Our aim is to find an expression for $\mathbf{r}(t)$. Let's fix t . One viewer needs to see the point $\mathbf{r}(t)$ and think she's seeing the point $A = (t, f(t), 0)$. Thus, the line through A parallel to the vector $\langle 0, 1, 1 \rangle$ must pass through $\mathbf{r}(t)$. Likewise, for the second viewer to see $\mathbf{r}(t)$ and think he's seeing $B = (t, g(t), 0)$, the line through B parallel to the vector $\langle 0, -1, 1 \rangle$ must pass through $\mathbf{r}(t)$. In other words, $\mathbf{r}(t)$ is the intersection point of these two lines.

It is straightforward to show that $\mathbf{r}_A(s) = (t, s + f(t), s)$ and $\mathbf{r}_B(s) = (t, -s + g(t), s)$ are the parameterizations of the lines through A and B , respectively. Moreover, the point of intersection is $(t, \frac{1}{2}(f(t) + g(t)), \frac{1}{2}(g(t) - f(t)))$. In particular, the parameterization of our curve is

$$\mathbf{r}(t) = \left(t, \frac{1}{2}(f(t) + g(t)), \frac{1}{2}(g(t) - f(t))\right).$$

We may now plug in our functions. The portion of our curve with nonnegative y -coordinates is given by

$$\mathbf{r}_1(t) = \left(t, \frac{1}{2}(1 - |t| + \sqrt{1 - t^2}), \frac{1}{2}(\sqrt{1 - t^2} + |t| - 1)\right)$$

for $-1 \leq t \leq 1$, and the other half is given by

$$\mathbf{r}_2(t) = \left(t, \frac{1}{2}(|t| - 1 - \sqrt{1 - t^2}), \frac{1}{2}(1 - \sqrt{1 - t^2} - |t|)\right)$$

for $-1 \leq t \leq 1$.

We encourage the reader to use three-dimensional

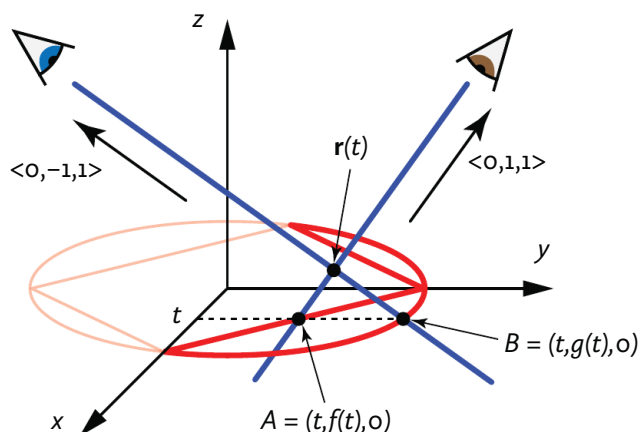


Figure 4. The point $\mathbf{r}(t)$ lies at the intersection of lines through A and B ; both lines are in the plane $x = t$.

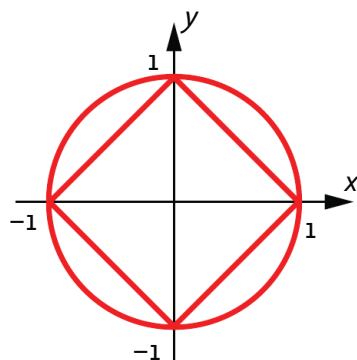


Figure 3. Two closed curves in the xy -plane.

For a look at Kokichi Sugihara's work, visit his web page <http://bit.ly/29isMcS> and check out his 2015 book, *Joy of Ambiguous Solids: How to Make Anomalous Objects That Change Their Appearances in a Mirror*.

graphing software—or better, a 3D printer—to create her own impossible shapes using other functions f and g .

A Paper Impossible Cylinder

The impossible cylinder is almost unbelievable—a picture doesn't do it justice. Thus, the reader may wish to make one from paper. Download the printable pdf (shown in figure 5) from maa.org/mathhorizons/supplemental.htm, and follow these simple instructions.

1) Cut out the shape at the top of the page.

2) Fold a sharp crease along the dotted line.

3) Tape the left and right edges together.

4) Fold a sharp crease along the taped seam.

5) Gently squeeze together the creased sides so that the shape opens. Looking down on the cylinder, it should have the shape at the bottom of the printout.

6) Close one eye. Look down on the shape at a 45-degree angle so that the two creases line up with each other.

7) Then turn it around 180 degrees and look again.

Making the template was an interesting mathematical challenge. It required parameterizing the lower shape in figure 5 by arclength, which had to be done numerically. We leave that as an exercise to the reader. ■

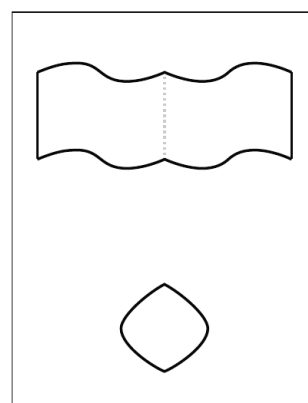


Figure 5. A printable template.

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