CHAPTER 3

Topological properties: Hausdorff, connected, and compact

1. Hausdorff spaces

In this chapter we begin to find tools that will help us in the classification problem. We will study two properties that a given topological space may or may not possess. We will see that two spaces that are homeomorphic will either both possess or both not possess the property. Thus we may be able to distinguish two spaces that are not homeomorphic.

Our first property states that any two points can be separated by disjoint open sets.

DEFINITION 3.1. A topological space X is *Hausdorff* iff for any $x, y \in X$ such that $x \neq y$ there exist open sets $U, V \subset X$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

THEOREM 3.2. *If X has the discrete topology, then X* *is/is not* *Hausdorff.*

PROOF.

Prove. □

THEOREM 3.3. *If X has at least two points and has the trivial topology, then X* *is/is not* *Hausdorff.*

PROOF.

Prove. □

EXAMPLE 3.4. Let $X = \{a, b, c\}$

List all the Hausdorff topologies of X. You may wish to go back and look at Exercise 1.9. Justify your answers.

THEOREM 3.5.

State and prove a theorem about finite Hausdorff spaces.

THEOREM 3.6. $\mathbf{E}^1$ *is/is not* *Hausdorff.*