January 31, 2007

Calculus II : Prime suspect

SOLUTIONS

An integer $p > 1$ is *prime* if its only positive divisors are 1 and p. In his 300 BC masterpiece *Elements* Euclid proved that there are infinitely many prime numbers. If you look at a list of all prime numbers you will not see an obvious pattern. There are many (we believe infinitely many) *twin primes*—pairs of prime numbers that are two numbers apart—such as 3 and 5, 11 and 13, and 1783820163777864646937291986073621 and 1783820163777864646937291986073623 (seriously!).[1] There are also *prime deserts*—prime-free sequences of integers—of arbitrarily large length.

However, the number of primes do seem to grow with some sort of regular pattern. The purpose of this lab is to use calculus to find a function $p(x)$ that approximates the rate of growth of the prime numbers.

If we are able to accomplish this, then we will have achieved a remarkable feat, since the integers and the primes are discrete values, and calculus is the study of the continuous!

> Definition. Let $\pi(x)$ be the number of prime numbers less than or equal to x.

1. What are the following values?

   (a) $\pi(1)$

   0

   (b) $\pi(2)$

   1

   (c) $\pi(3)$

   2

   (d) $\pi(4)$

   2

   (e) $\pi(5.3)$

   3

   (f) $\pi(10)$

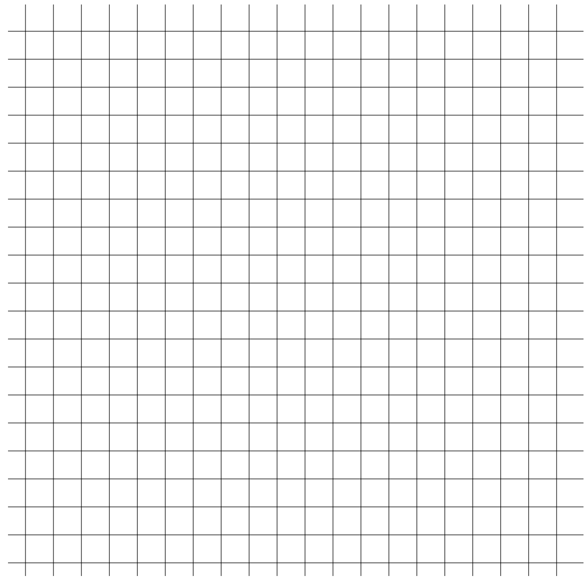
   4

   (g) $\pi(20)$

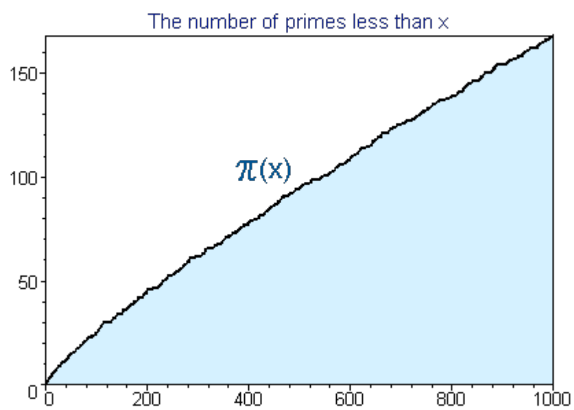
   8

[1] These twin primes were discovered by Dominic Klyve while visiting Dickinson College in 2007.

2. Sketch the graph of $y = \pi(x)$ on the interval $[0, 20]$.



Here is the graph of $y = \pi(x)$ on the interval $[0, 1000]$.



3. Before we tackle $\pi(x)$, we will take a brief detour to solve some seemingly unrelated problems.

   (a) What is the definition of $\ln x$?

$$\ln x = \int_1^x \frac{1}{t}\, dt, \quad (x > 0)$$

   (b) Suppose $g'(x) = \dfrac{1}{x}$ (for $x > 0$) and $g(1) = k$. What is $g(x)$?

$$g(x) = \ln x + k$$

(c) Suppose f is a differentiable function and $y = f(ax)$ where a is a constant. What is dy/dx?

$$\frac{dy}{dx} = af'(ax)$$

(d) Suppose f is a function, a and k are constants, and $f'(1) = 1$. Further, suppose that

$$f(ax) = f(x) + f(a) + k$$

for all x-values. What is $f'(a)$? (Hint: take the derivative of this equation.)

$$\begin{aligned} af'(ax) &= f'(x) \\ af'(a) &= f'(1) = 1 \\ f'(a) &= \frac{1}{a} \end{aligned}$$

(e) Suppose f is a function, k is a constant, $f'(1) = 1$, and

$$f(ax) = f(x) + f(a) + k$$

for **all** a and all x. What is $f'(x)$? If, in addition, $f(1) = k$, what is $f(x)$?

$$\begin{aligned} f'(x) &= \frac{1}{x} \\ f(x) &= \ln x + k \end{aligned}$$

The following chart lists the number of primes less than all of the powers of 10, up to 10^{23}.[2] The third column is $\pi(x)/x$—the fraction of numbers up to x that are prime. The fourth column is the reciprocal of the third column.

x	$\pi(x)$	$\pi(x)/x$	$x/\pi(x)$
10	4	.4	2.5
100	25	.25	4
$1,000$	168	.168	5.95238
10^4	$1,229$	.1229	8.1367
10^5	$9,592$	.09592	10.4254
10^6	$78,498$	.078498	12.7392
10^7	$664,579$	.0664579	15.0471
10^8	$5,761,455$	.05761455	17.3567
10^9	$50,847,534$	.050847534	19.6666
10^{10}	$455,052,511$	.0455052511	21.97548581
10^{11}	$4,118,054,813$	.0411805481	24.28330961
10^{12}	$37,607,912,018$	.0376079120	26.59014942
10^{13}	$346,065,536,839$	.0346065536	28.89626078
10^{14}	$3,204,941,750,802$	.0320494175	31.20181512
10^{15}	$29,844,570,422,669$	.0298445704	33.50693228
10^{16}	$279,238,341,033,925$	.0279238341	35.81170109
10^{17}	$2,623,557,157,654,233$	.0262355715	38.11618881
10^{18}	$24,739,954,287,740,860$	.0247399542	40.42044655
10^{19}	$234,057,667,276,344,607$	.0234057667	42.72451364
10^{20}	$2,220,819,602,560,918,840$	.0222081960	45.02842098
10^{21}	$21,127,269,486,018,731,928$	.0211272694	47.33219314
10^{22}	$201,467,286,689,315,906,290$	.0201467286	49.63584989
10^{23}	$1,925,320,391,606,803,968,923$	.0192532039	51.93940729

[2] In case you're wondering, the last line of the table says that there are one sextillion, nine hundred twenty-five quintillion, three hundred twenty quadrillion, three hundred ninety-one trillion, six hundred six billion, eight hundred three million, nine hundred sixty-eight thousand, nine hundred twenty-three prime numbers less than or equal to one hundred sextillion—approximately 1.9%.

4. Our first task is to find a function $f(x)$ that approximates $x/\pi(x)$ (the fourth column) for large values of x.

   (a) Find an approximate relationship between $f(10^{11})$, $f(10^{12})$, and $f(10^{23})$.

$$f(10^{23}) = f(10^{11}) + f(10^{12}) + 1.065\ldots$$

   (b) Find an approximate relationship between $f(10^{8})$, $f(10^{14})$, and $f(10^{22})$.

$$f(10^{22}) = f(10^{8}) + f(10^{14}) + 1.077\ldots$$

   (c) Use (a) and (b) to find a simple formula for $f(ab)$.

$$f(ab) = f(a) + f(b) + 1.07$$

(d) Assume that $f'(1) = 1$ and $f(1) = k$ (k is the constant from 4(c)). Use this information together with your answers from 4(c) and 3(e) to find $f(x)$.

$$f(x) = \ln x + 1.07$$

5. In the previous problem you found a function $f(x)$ that is approximately the same as $x/\pi(x)$. Use this relationship to find an approximation for $\pi(x)$—call it $p(x)$.

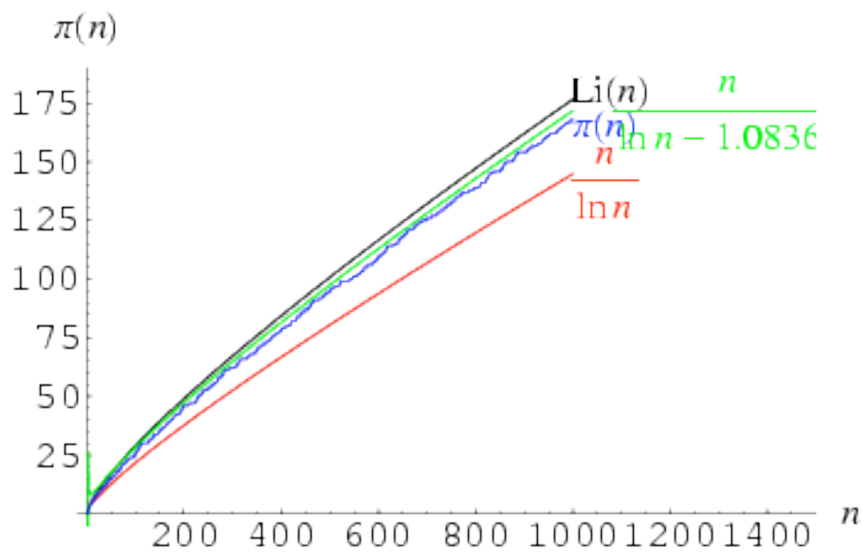
$$p(x) = \frac{x}{\ln x + 1.07}$$

6. Use Maple to compute $p(10)$, $p(100)$, $p(10^{12})$, and $p(10^{23})$.

x	$\pi(x)$	$p(x)$
10	4	2.97
100	25	17.62
10^{12}	$37,607,912,018$	3.48×10^{10}
10^{23}	$1,925,320,391,606,803,968,923$	1.850×10^{21}

How well did we do? Would you say that $p(x)$ gives a good approximation of $\pi(x)$?

Below is a graph of $y = \pi(x)$ along with three different approximations—the one you computed $y = p(x)$ (this was originally discovered by Legendre), $y = x/\ln x$, and a function named $Li(x)$ (which was discovered by Gauss).[3]



Although none of these functions follow the curve of $y = \pi(x)$ exactly, they all come very close. In fact, they all **grow at the same rate** as $\pi(x)$. That is, the ratios $\pi(x)/p(x)$, $\pi(x)/\frac{x}{\ln x}$, and $\pi(x)/Li(x)$ all tend to 1 as x goes to infinity. Legendre and Gauss discovered their functions in the late eighteenth or early nineteenth centuries, but a rigorous proof that these were good approximations came a century later, in 1896, by Hadamard and Vallée Poussin. This important theorem is now called the *prime number theorem*.

The prime number theorem. Let $\pi(x)$ be the number of prime numbers less than or equal to x, then $\displaystyle\lim_{x\to\infty} \frac{\pi(x)}{p(x)} = 1$.

[3]In case you were wondering, $y = Li(x) = \displaystyle\int_2^x \frac{1}{\ln t}\,dt = C + \frac{x}{\ln x} + \frac{x}{(\ln x)^2} + \frac{x}{(\ln x)^3} + \cdots$.

7. Compute $\dfrac{\pi(x)}{p(x)}$ for $x = 10, 100, 10^{12}, 10^{23}$.

x	$\pi(x)$	$\pi(x)/p(x)$
10	4	1.349
100	25	1.419
10^{12}	$37,607,912,018$	1.079
10^{23}	$1,925,320,391,606,803,968,923$	1.040

We still do not fully understand the distribution of the prime numbers. In fact, the most famous unsolved problem in mathematics, the *Riemann Hypothesis*, is closely related to the prime number theorem. The first person to prove the Riemann Hypothesis will be awarded **$1 million** by the Clay Mathematics Institute (in addition to getting her or his picture on the front page of every major news publication).